Expressing Flexibility in Logic Synthesis by Boolean Relations

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Overview

- Classic applications
  - Multilevel logic optimization
- Approximate Logic Synthesis
- Bounded-depth logic synthesis via Boolean relations
  - Bi-decomposed Circuits
  - Synthesis with critical signals: P-circuits

Multilevel logic optimization

Given a multilevel logic network, obtain an **equivalent** representation of the network, **optimal** w.r.t. a cost function involving area and delay

- *identifying subnetworks* to be optimized,
- deriving their *flexibility*
- and *replacing* such subnetworks by *simpler, optimized* ones

Single-output subnetworks

The flexibility for implementing the node's function can be represented by don't cares

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Multi-output subnetworks

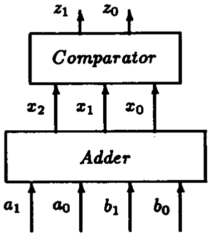
Don't cares are not sufficient for representing all the flexibility

> Don't care-based methods allow us to optimize only *one single-output subnetwork at a time*

Boolean relations describe all the flexibility

> *Boolean relations allow the simultaneous modification of all nodes of a subnetwork*

Minimization of a *two-bit adder* due to the filtering effect of a *comparator* [Brayton, Somenzi, 1989]



$z = 01 \Rightarrow a + b < 3$

$z = 00 \Rightarrow (a + b = 3) \vee (a + b = 4)$

$z = 10 \Rightarrow a + b > 4.$

Input values can be partitioned into three equivalence classes:

Values less than 3: {000, 001, 010}
Values equal to 3 or 4: {011, 100}
Values greater than 4: {101, 110, 111}

The values in each class are not distinghished by the comparator

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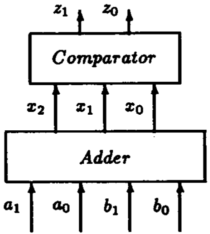

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We can change the output value of the adder, to any other value in the same equivalence class.

Boolean relation describing the flexible adder

$a_1\, a_0\, b_1\, b_0$	$x_2\, x_1\, x_0$
$\{0000, 0001, 0010, 0100, 1000, 0101\}$	$\{000, 001, 010\}$
$\{0011, 0110, 1001, 1010, 1100, 0111, 1101\}$	$\{011, 100\}$
$\{1011, 1110, 1111\}$	$\{101, 110, 111\}$

Minimization of a *two-bit adder* due to the filtering effect of a *comparator* [Brayton, Somenzi, 1989]

Minimization with don't cares

including the normal adder as an acceptable implementation

$a_1\ a_0\ b_1\ b_0$	$x_2\ x_1\ x_0$
1 1 - 0	0 1 1
- 1 1 0	0 1 1
1 0 - 1	0 1 1
- 0 1 1	0 1 1
- 1 1 1	1 0 0
1 1 - 1	1 0 0
1 1 1 -	0 1 0
1 - 1 -	1 0 0

Minimization of the Boolean relation

much simpler minimum solution

$a_1\ a_0\ b_1\ b_0$	$x_2\ x_1\ x_0$
0 - 1 -	0 1 0
1 - 0 -	0 1 0
1 - 1 -	1 0 0
- - - 1	0 0 1
- 1 - -	0 0 1

Approximate Logic Synthesis (ALS)

- Exploit error tolerance of applications to implement approximate designs with
  - smaller area
  - smaller delay
  - or lower energy consumption
- Modify some outputs of a function, so that the produced error is tolerable

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Error frequency

number of minterms on which an error occurs, as a fraction of the total number of minterms

Error magnitude

maximum amount by which the numerical value at the outputs of a function can deviate from the exact value

ALS and Boolean Relations

[Miao, Gerstlauer, Orshansky, 2013]:

- ALS under arbitrary *error magnitude* and *error frequency* constraints
- Two-level logic minimization algorithm, two-phase approach:

1. derive the solution of the problem constrained only by the *magnitude of errors*

2. the solution is iteratively refined to meet the original *error frequency constraint*

ALS and Boolean Relations

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- ALS under arbitrary *error magnitude* and *error frequency* constraints
- Two-level logic minimization algorithm, two-phase approach:

1. derive the solution of the problem constrained only by the *magnitude of errors* → expressed and solved using Boolean relations

2. the solution is iteratively refined to meet the original *error frequency constraint*

ALS constrained by Error Magnitude only

- Multi-output function $f : \{0,1\}^n \rightarrow \{0,1\}^k$
- M: constrain on the magnitude of possible errors

PROBLEM

Find $f' : \{0,1\}^n \rightarrow \{0,1\}^k$ of **minimal cost** s.t.

$$\forall x \in \{0,1\}^n \quad |f(x) - f'(x)| \leq M$$

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$$\forall x \in \{0,1\}^n \quad |f(x) - f'(x)| \leq M$$

- $\forall x \in \{0,1\}^n$, $e(x) =$ output error set for x

  additional values that the function can take while satisfying the error magnitude constraint

$$\mathcal{R}_{f'}(x) \in \{f(x) \cup e(x)\}$$

each input corresponds to more than one output: f' becomes a Boolean relation $\mathcal{R}_{f'}$

$\Rightarrow$ **minimize the Boolean relation $\mathcal{R}_{f'}$**, under a given metric (i.e., the number of literals in a SOP representation)

Example

ADDER

$x_1\ x_2$	$f(x_1, x_2)$
0 0	00
0 1	01
1 0	01
1 1	10

$$L(f) = 6$$

$$SOP(f^{(1)}) = x_1 x_2$$
$$SOP(f^{(2)}) = \overline{x}_1 x_2 + x_1 \overline{x}_2$$

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ADDER, $M = 1$

$x_1\, x_2$	$\mathcal{R}_{f'}(x_1, x_2)$
0 0	$\{00, 01\}$
0 1	$\{01, 00, 10\}$
1 0	$\{01, 00, 10\}$
1 1	$\{10, 01, 11\}$

Example

ADDER

$x_1\ x_2$	$f(x_1, x_2)$
0 0	00
0 1	01
1 0	01
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0 1	$\{01, 00, 10\}$
1 0	$\{01, 00, 10\}$
1 1	$\{10, 01, 11\}$

$$L(f') = 0$$

$$SOP(f^{(1)}) = 0$$
$$SOP(f^{(2)}) = 1$$

Example

ADDER

$x_1\ x_2$	$f(x_1, x_2)$
0 0	00
0 1	01
1 0	01
1 1	10

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ADDER, $M = 1$

$x_1\ x_2$	$\mathcal{R}_{f'}(x_1, x_2)$	
0 0	$\{00, 01\}$	X
0 1	$\{01, 00, 10\}$	
1 0	$\{01, 00, 10\}$	
1 1	$\{10, 01, 11\}$	X

$L(f') = 0$

$SOP(f^{(1)}) = 0$
$SOP(f^{(2)}) = 1$

ERROR FREQUENCY: 50 %

Frequency constrained ALS algorithm

- The BR solution may not satisfy the **constrain on error frequency**
- Iterative and greedy algorithm for **systematically corrects the wrong outputs** (leading to the smallest cost increase) **until the error frequency constraint is met**

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Error frequency: 25 %

$x_1\ x_2$	$f(x_1, x_2)$	$\mathcal{R}_{f'}(x_1, x_2)$	
0 0	00	$\{00, 01\}$	X
0 1	01	$\{01, 00, 10\}$	
1 0	01	$\{01, 00, 10\}$	
1 1	10	$\{10, 01, 11\}$	X

Frequency constrained ALS algorithm

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1 0	01	$\{01, 00, 10\}$
1 1	10	$\{10, 01, 11\}$ X

$$L(f') = 2$$

$$SOP(f^{(1)}) = 0$$
$$SOP(f^{(2)}) = x_1 + x_2$$

Bounded-depth logic synthesis via Boolean relations: Bi-Decomposition (joint work with R. K. Brayton, V.Ciriani, G. Trucco, and T. Villa)

$f : \{0,1\}^n \to \{0,1,-\}$

$f = (f_{on}, f_{dc}, f_{off})$ can be covered with

→ a SOP derived by the on-set (+ some dc-points)

→ a POS resulting from the complement of a SOP for the off-set (+ some dc-points)

⋆ *Which one gives the best cover, the SOP or the POS form?*

⋆ *Can we study a form that is part in SOP and part in POS form, and is better than both?*

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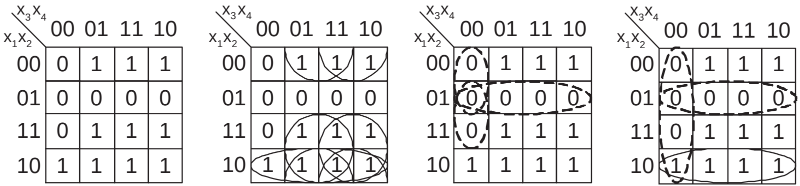
- $\rightarrow$ a POS resulting from the complement of a SOP for the off-set (+ some dc-points)
  - ⋆ *Which one gives the best cover, the SOP or the POS form?*
  - ⋆ *Can we study a form that is part in SOP and part in POS form, and is better than both?*

We propose a **bi-decomposed form** that is part in SOP form and part in POS

$$f_B = \overline{f_0} \ \text{op} \ f_1$$

Bounded-depth logic synthesis via Boolean relations: Bi-Decomposition



(a) f (b) SOP for f (c) POS for f (d) Bi-cond. form for f

- $f_{SOP} = f_1^{SOP} = x_1\overline{x}_2 + x_1x_3 + x_1x_4 + \overline{x}_2x_3 + \overline{x}_2x_4$ 10 literals
- $f_{POS} = \overline{f}_0^{POS} = (x_1 + \overline{x}_2)(x_1 + x_3 + x_4)(\overline{x}_2 + x_3 + x_4)$ 8 literals

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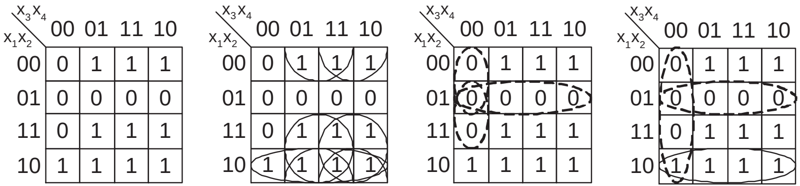

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- $f_B = \overline{f}_0 + f_1 = ((x_1 + \overline{x}_2)(x_3 + x_4)) + x_1\overline{x}_2$ 6 literals

1000 is in the the OFF set of $\overline{f}_0$ and in the ON set of f_1
thus is in the ON set of the $\overline{f}_0 + f_1$

Synthesis of Bi-decomposed Circuits and Boolean relations

$f : \{0,1\}^n \to \{0,1,-\}$, $f = u \; \mathbf{op} \; v$ $u \leftarrow \overline{f_0},\; v \leftarrow f_1$

Inputs of u and v: the same as the inputs of f: $x_1, \ldots, x_n$
Output: is the output that **op** takes on $u(x_1, \ldots, x_n)$ and $v(x_1, \ldots, x_n)$

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AND GROUP

$u\,v$	AND
$\overline{u}\,v$	$\not\Leftarrow$
$u\,\overline{v}$	$\not\Rightarrow$
$\overline{u}\,\overline{v}$	NOR

OR GROUP

$u + v$	OR
$\overline{u} + v$	$\Rightarrow$
$u + \overline{v}$	$\Leftarrow$
$\overline{u} + \overline{v}$	NAND

XOR GROUP

$u \oplus v$	XOR
$u \ \overline{\oplus}\ v$	XNOR

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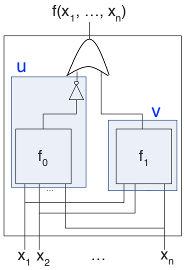
$u \overline{\oplus} v$	XNOR

- The two-input operator induces flexibility that cannot be expressed exactly by don't care conditions only: **a Boolean relation is required**
- For each binary **op**, we define $\mathcal{R}_{op} : \{0,1\}^n \rightarrow \{0,1\}^2$ s.t.
  - the set of functions compatible with $\mathcal{R}_{op}$ corresponds to the set of pairs (u, v) occurring in all bi-decomposed circuit implementations of f w.r.t. op.
  - an **optimal solution of $\mathcal{R}_{op}$** is an **optimal bi-decomposed circuit** for f

Construction of $\mathcal{R}_{OR}$



- $\forall x \in f_{on}$, x must be associated to one of the three output values on which $u + v$ evaluates to 1

$$\mathcal{R}_{OR}(x) = \{01, 10, 11\} = \{1-, -1\}$$

- $\forall x \in f_{off}$, x must be associated to the output 00 on which $u + v$ evaluates to 0

$$\mathcal{R}_{OR}(x) = 00$$

- $\forall x \in f_{dc}$, x can be associated to any output

$$\mathcal{R}_{OR}(x) = \{--\}$$

Construction of $\mathcal{R}_{op}$

AND table

	$\mathcal{R}_{AND}$	$\mathcal{R}_{\not\Leftarrow}$	$\mathcal{R}_{NOR}$	$\mathcal{R}_{\not\Rightarrow}$
$x \in f_{on}$	$\{11\}$	$\{01\}$	$\{00\}$	$\{10\}$
$x \in f_{off}$	$\{0-, -0\}$	$\{1-, -0\}$	$\{1-, -1\}$	$\{0-, -1\}$
$x \in f_{dc}$	$\{--\}$	$\{--\}$	$\{--\}$	$\{--\}$

OR table

	$\mathcal{R}_{OR}$	$\mathcal{R}_{\Rightarrow}$	$\mathcal{R}_{NAND}$	$\mathcal{R}_{\Leftarrow}$
$x \in f_{on}$	$\{1-, -1\}$	$\{0-, -1\}$	$\{0-, -0\}$	$\{1-, -0\}$
$x \in f_{off}$	$\{00\}$	$\{10\}$	$\{11\}$	$\{01\}$
$x \in f_{dc}$	$\{--\}$	$\{--\}$	$\{--\}$	$\{--\}$

XOR table

	$\mathcal{R}_{XNOR}$	$\mathcal{R}_{XOR}$
$x \in f_{on}$	$\{00, 11\}$	$\{01, 10\}$
$x \in f_{off}$	$\{01, 10\}$	$\{00, 11\}$
$x \in f_{dc}$	$\{--\}$	$\{--\}$

Construction of $\mathcal{R}_{op}$

The three tables are distinguished by whether

- the **offset** is partitioned **(AND group)**
- the **onset** is partitioned **(OR group)**
- or **both** are partitioned **(XOR group)**

How this partitioning is done is task of the *Boolean relation minimizer*

Construction of $\mathcal{R}_{op}$

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- or **both** are partitioned **(XOR group)**

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Bi-decomposed circuit minimization problem

$\Longleftrightarrow$

problem of finding an optimal implementation of $\mathcal{R}_{op}$

Good gains in a majority of benchmarks against affordable increases in synthesis time

Synthesis with critical signals: P-circuits
(joint work with V. Ciriani, G. Trucco, and T. Villa)

Scenario

- Logic synthesis in presence of critical signals that should be moved toward the output
- signals with high switching activity
  $\rightarrow$ for decreasing power consumption
- late arriving signals
  $\rightarrow$ for decreasing circuit delay

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Problem

Restructure (or synthesize) a circuit in order to move critical signals near to the output (decreasing the cone of influence)

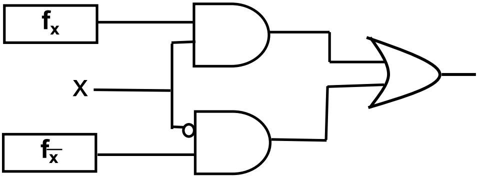
- **minimizing** the circuit **area**
- keeping the number of levels **bounded**
- performing an **efficient** minimization

Simple solution: Shannon decomposition

- x is the critical signal

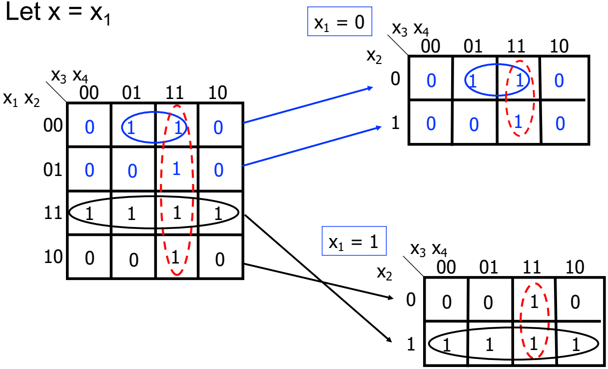
$$f = \overline{x}\, f_{|x=0} + x\, f_{|x=1}$$

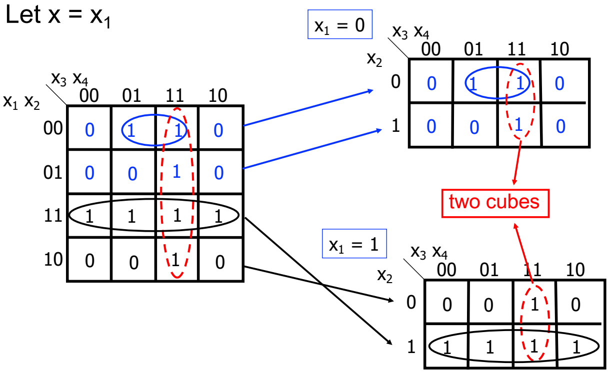
- the cofactors $f_{|x=0}$ and $f_{|x=1}$ do not depend on x
- x is near to the output



Problem of Shannon approach

Let $x = x_1$





It is not a compact representation

Decomposition with intersection

- try not to split the cubes
- let the critical signal near to the output

IDEA

- the cubes that **do not depend on x** and cross the two sets are not projected
- how to identify these cubes?

Decomposition with intersection

- try not to split the cubes
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Idea

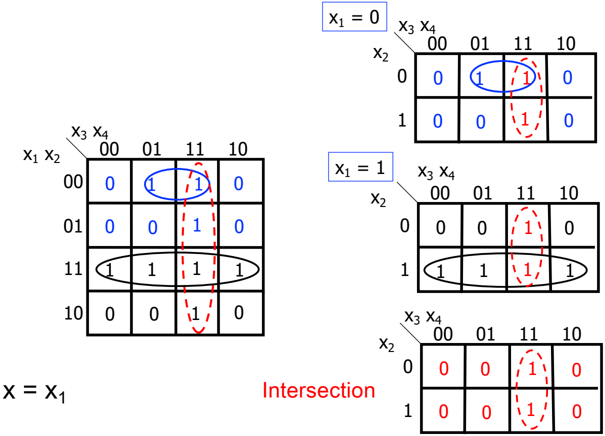
- the cubes that **do not depend on x** and cross the two sets are not projected
- how to identify these cubes?

  They are in the **intersection between the two cofactors**

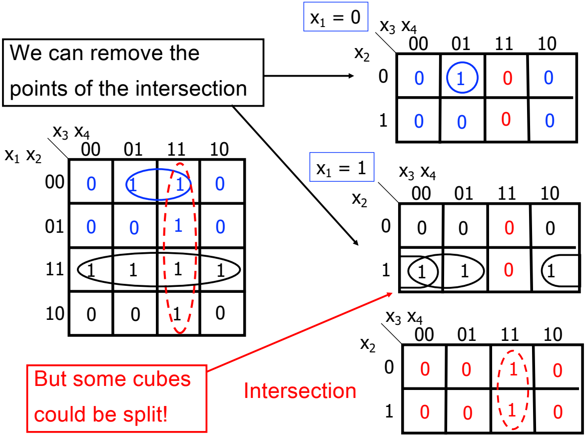
$$I = f_{|x_i=0} \cap f_{|x_i=1}$$

- keep I **unprojected**, and project only the minterms in $f_{|x_1=0} \setminus I$ and $f_{|x_i=1} \setminus I$

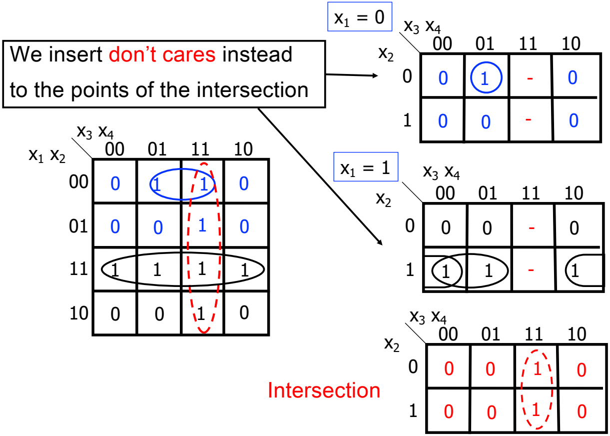
Example


x3 x4
x1 x2
00 01 11 10
00 0 1 1 0
01 0 0 1 0
11 1 1 1 1
10 0 0 1 0
x = x1
x1 = 0
x3 x4
x2
00 01 11 10
0 0 1 1 0
1 0 0 1 0
x1 = 1
x3 x4
x2
00 01 11 10
0 0 0 1 0
1 1 1 1 1
Intersection
x3 x4
x2
00 01 11 10
0 0 0 1 0
1 0 0 1 0

Example


We can remove the points of the intersection
But some cubes could be split!
Intersection
x1 = 0
x1 = 1
x3 x4
x1 x2
x2
00 01 11 10
00 0 1 1 0
01 0 0 1 0
11 1 1 1 1
10 0 0 1 0
0 0 1 0 0
1 0 0 0 0
0 0 0 0 0
1 1 1 0 1
0 0 0 1 0
1 0 0 1 0

Example


We insert don't cares instead
to the points of the intersection
x3 x4
x1 x2
00 01 11 10
00 0 1 1 0
01 0 0 1 0
11 1 1 1 1
10 0 0 1 0
x1 = 0
x3 x4
x2
00 01 11 10
0 0 1 - 0
1 0 0 - 0
x1 = 1
x3 x4
x2
00 01 11 10
0 0 0 - 0
1 1 1 - 1
x3 x4
x2
00 01 11 10
0 0 0 1 0
1 0 0 1 0
Intersection

Example

Decomposition with intersection

$x_1 x_2$ \ $x_3 x_4$	00	01	11	10
00	0	1	1	0
01	0	0	1	0
11	1	1	1	1
10	0	0	1	0

The cubes are not split

$x_1 = 0$

x_2 \ $x_3 x_4$	00	01	11	10
0	0	1	1	0
1	0	0	0	0

$x_1 = 1$

x_2 \ $x_3 x_4$	00	01	11	10
0	0	0	0	0
1	1	1	1	1

Intersection

x_2 \ $x_3 x_4$	00	01	11	10
0	0	0	1	0
1	0	0	1	0

P-circuits for completely specified functions

- if a **point is in** I and is useful for a better minimization of $f_{|x_i=0}$ and $f_{|x_i=1}$, **it can be kept both in the cofactors and in the intersection**
- if a **point is covered in both the projected cofactors**, it is **not necessary to cover it in** I (replaced by a don't care in I)

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P-circuit

A *P-circuit* of a completely specified function f is the circuit

$$P(f) = \overline{x}_i\, f^{=} + x_i\, f^{\neq} + f^{I}$$

1. $I = f_{|x_i=0} \cap f_{|x_i=1}$
2. $(f_{|x_i=0} \setminus I) \subseteq f^{=} \subseteq f_{|x_i=0}$
3. $(f_{|x_i=1} \setminus I) \subseteq f^{\neq} \subseteq f_{|x_i=1}$
4. $\emptyset \subseteq f^{I} \subseteq I$
5. $P(f) = f$

Minimization of P-circuits using Boolean Relation

Find the sets $f^{=}, f^{\neq}, f^{I}$ leading to a **P-circuit of minimal cost**

- $f : \{0,1\}^n \to \{0,1\}$ $\mathcal{R}_f : \{0,1\}^{n-1} \to \{0,1\}^3$
- **Input set for $\mathcal{R}_f$**: all input variables but the critical signal x_i
- **Output set for $\mathcal{R}_f$**: all triple of functions $f^{=}, f^{\neq}, f^{I}$ defining a P-circuit for f

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$x_1 \dots x_{i-1} x_{i+1} \dots x_n$	$\mathcal{R}_f = (f^{=}, f^{\neq}, f^{I})$
points in $f_{\vert x_i=0} \setminus I$	$\{100\}$
points in $f_{\vert x_i=1} \setminus I$	$\{010\}$
points in I	$\{--1, 11-\}$
all other points	$\{000\}$

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Theorem

P-circuit minimization for f

$\Longleftrightarrow$

minimization of the Boolean relation $\mathcal{R}_f$

P-circuits minimized with Boolean relations are more compact than P-circuits expressed and minimized as incompletely specified functions

Conclusions

- **Boolean relations can be extremely useful for modeling Boolean hard optimization problems**
  - → with Boolean relations we can model problems that cannot be completely described with incompletely specified functions
- **Problem**: scalability of the approach
  - Boolean relation minimization is a very hard problem
  - Boolean relation minimizers cannot handle relations with many outputs
- Boolean relations have been successfully used in logic synthesis to solve problems that can be cast as the minimization of *relations with a constant number of outputs* (2, 3)

THANK YOU